A STRONG VERSION OF THE SIMS CONJECTURE ON FINITE PRIMITIVE PERMUTATION GROUPS

Anatoly Kondrat'ev, Vladimir Trofimov

Krasovskii Institute of Mathematics and Mechanics,
Ekaterinburg, Russia
E-mail: a.s.kondratiev@imm.uran.ru, trofimov@imm.uran.ru

Ischia Group Theory 2018, Ischia (Naples, Italy), March 2018

Let G be a primitive permutation group on a finite set X and $x \in X$. Let d be the length of some G_x-orbit on $X \setminus \{x\}$. It is easy to see that $d = 1$ implies $G_x = 1$ (and $G \cong \mathbb{Z}_p$ for a prime p) and $d = 2$ implies $G_x \cong \mathbb{Z}_2$ (and $G \cong D_{2p}$ for an odd prime p). In [Math. Z. 95 (1967)], Ch. Sims adapted arguments by W. Tutte concerning vertex stabilizers of cubic (i.e. of valency 3) graphs in vertex-transitive groups of automorphisms (see [Proc. Camb. Phil. Soc. 43 (1947)] and [Canad. J. Math. 11 (1959)]) to prove that $d = 3$ implies $|G_x|$ divides $3 \cdot 2^4$. In connection with this result Ch. Sims made the following general conjecture which is now well known as the Sims conjecture.

SIMS CONJECTURE. There exists a function $f : \mathbb{N} \to \mathbb{N}$ such that, if G is a primitive permutation group on a finite set X, G_x is the stabilizer in G of a point x from X, and d is the length of some non-trivial G_x-orbit on $X \setminus \{x\}$, then $|G_x| \leq f(d)$.

Some progress toward proving this conjecture had been obtained in papers of Sims (Math. Z. 95 (1967)), Thompson (J. Algebra 14 (1970)), Wielandt (Ohio State Univ. Lecture Notes, 1971), Knapp (Math. Z. 133 (1973), Arch. Math. 36 (1981)), Fomin (In: Sixth All-Union Symp. on Group Theory, Naukova Dumka, Kiev, 1980). So, Thompson and independently Wielandt proved that $|G_x/O_p(G_x)|$ is bounded by some function of d for some prime p. Fomin proved that $|G_x|$ is bounded by some function of the maximal length of G_x-orbits. But only with the use of the classification of finite simple groups, the validity of the conjecture was proved by Cameron, Praeger, Saxl and Seitz (Bull. London Math. Soc. 15 (1983)).

The Sims conjecture can be formulated in terms of graphs as follows.

For an undirected connected graph Γ (without loops or multiple edges) with vertex set $V(\Gamma)$, $G \leq Aut(\Gamma)$, $x \in V(\Gamma)$, and $i \in \mathbb{N} \cup \{0\}$, we will denote by $G_x^{[i]}$ the elementwise stabilizer in G of the (closed) ball of radius i of the graph Γ centered at x in the natural metric d_Γ on $V(\Gamma)$.

Let G be a primitive permutation group on a finite set X, $|X| > 1$, $x \in X$, and $M = G_x$. Fix an element $a \in G$ with $a(x) \neq x$. Consider the graph Γ with vertex set $V(\Gamma) = X$ and edge set $E(\Gamma) = \{\{g(x),\ ga(x)\} \mid g \in G\}$. Then Γ is an undirected connected graph, G is an automorphism group of Γ acting primitively on $V(\Gamma)$, and the length of the M-orbit containing $a(x)$ is equal either to the valency of Γ (if there exists an element in G that transposes x and $a(x)$) or to the half of the valency of Γ (otherwise). Therefore, the Sims conjecture is equivalent to the following statement.

SIMS CONJECTURE (GEOMETRICAL FORM). There exists a function $\psi : \mathbb{N} \cup \{0\} \longrightarrow \mathbb{N}$ such that, if Γ is an undirected connected finite graph and G is its automorphism group acting primitively on $V(\Gamma)$, then $G_x^{[\psi(d)]} = 1$ for $x \in V(\Gamma)$, where d is the valency of the graph Γ.

Using the classification of finite simple groups, the authors obtained in (Dokl. Math. 59 (1999)) the following result, which establishes the validity of a strengthened version of the Sims conjecture.

THEOREM 1. **If Γ is an undirected connected finite graph and G its automorphism group acting primitively on $V(\Gamma)$, then $G_x^{[6]} = 1$ for $x \in V(\Gamma)$.**

In other words, automorphisms of connected finite graphs with vertex-primitive automorphism groups are determined by images of vertices of any ball of radius 6.

Actually, we proved a result which is stronger than Theorem 1 (Theorem 2 below). It is formulated in terms of subgroup structure of finite groups. To formulate the result, we need the following definitions.

Recall that, for a group G and $H \leq G$, the subgroup $H_G = \bigcap_{g \in G} gHg^{-1}$ is called the core of the subgroup H in G.

For a group G, its subgroups M_1 and M_2, and any $i \in \mathbb{N}$, let us define by induction subgroups $(M_1, M_2)^i$ and $(M_2, M_1)^i$ of $M_1 \cap M_2$, which we will be called the ith mutual cores of M_1 with respect to M_2 and of M_2 with respect to M_1, respectively. Put

$$(M_1, M_2)^1 = (M_1 \cap M_2)_{M_1}, \quad (M_2, M_1)^1 = (M_1 \cap M_2)_{M_2}.$$

For $i \in \mathbb{N}$, assuming that $(M_1, M_2)^i$ and $(M_2, M_1)^i$ are already defined, put

$$(M_1, M_2)^{i+1} = ((M_1, M_2)^i \cap (M_2, M_1)^i)_{M_1},$$

$$(M_2, M_1)^{i+1} = ((M_1, M_2)^i \cap (M_2, M_1)^i)_{M_2}.$$

THEOREM 2. **Let G be a finite group, and let M_1 and M_2 be distinct conjugate maximal subgroups of G. Then, the subgroups $(M_1, M_2)^6$ and $(M_2, M_1)^6$ coincide and are normal in the group G.**

Under the hypothesis of Theorem 1 for $|V(\Gamma)| > 1$, if we set $M_1 = G_x$ and $M_2 = G_y$, where x and y are adjacent vertices of the graph Γ, then $G_x^{[i]} \leq (M_1, M_2)^i$ and $G_y^{[i]} \leq (M_2, M_1)^i$ for all $i \in \mathbb{N}$. Thus, Theorem 1 follows from Theorem 2.

The following result is also derived from Theorem 2.

Corollary. *Let G be a finite group, let M_1 be a maximal subgroup of G, and let M_2 be a subgroup of G containing $(M_1)_G$ and not contained in M_1. Then the subgroup $(M_1, M_2)^{12}$ coincides with $(M_1)_G$.*

We constructed also some examples which show that the constant 6 in Theorems 1 and 2 cannot be decreased, the constant 12 in Corollary cannot be decreased and the condition of maximality of the subgroup M_1 in G in Corollary is essential.

Theorem 2 immediately follows from a stronger result, which we are planning to prove in a series of papers.

Let G, M_1, and M_2 satisfy the hypothesis of Theorem 2. We are interested in the case where $(M_1)_G = (M_2)_G = 1$ and $1 < |(M_1, M_2)^2| \leq |(M_2, M_1)^2|$. The set of all such triples (G, M_1, M_2) is denoted by Π. Consider triples from Π up to the following equivalence: the triples (G, M_1, M_2) and (G', M_1', M_2') from Π are equivalent if there exists an isomorphism of G on G' taking M_1 to M_1' and M_2 to M_2'.

The group G acts by conjugation faithfully and primitively on the set $X = \{gM_1g^{-1} \mid g \in G\}$. According to a refinement of the Thompson–Wielandt theorem (1970) for the case under consideration the product $(M_1, M_2)^2(M_2, M_1)^2$ is a nontrivial p-group for some prime p.

Depending on the form of the socle $Soc(G)$ of the group G, we partition the set Π into the following subsets:

Π_0 is the set of triples (G, M_1, M_2) from Π such that $Soc(G)$ is not a simple nonabelian group, i.e., G is not an almost simple group;

Π_1 is the set of triples (G, M_1, M_2) from Π with $Soc(G)$ isomorphic to an alternating group;

Π_2 is the set of triples (G, M_1, M_2) from $\Pi \setminus \Pi_1$ with $Soc(G)$ isomorphic to a simple group of Lie type over a field of a characteristic different from p;

Π_3 is the set of triples (G, M_1, M_2) from $\Pi \setminus (\Pi_1 \cup \Pi_2)$ with simple $Soc(G)$ isomorphic to a simple group of Lie type over a field of characteristic p;

Π_4 is the set of triples (G, M_1, M_2) from Π with $Soc(G)$ isomorphic to one of the 26 finite simple sporadic groups.

For a nonempty set Σ of triples (G, M_1, M_2), where G is a finite group and M_1 and M_2 are distinct conjugate maximal subgroups of G, define $c(\Sigma)$ to be the maximum positive integer c such that $(M_1, M_2)^{c-1} \neq 1$ or $(M_2, M_1)^{c-1} \neq 1$ for some triple $(G, M_1, M_2) \in \Sigma$. If such a maximum number does not exist, we set $c(\Sigma) = \infty$. Define $c(G, M_1, M_2) = c(\{(G, M_1, M_2)\})$ and $c(\varnothing) = 0$.

It was announced in (Dokl. Math. 59 (1999)) that $c(\Pi_0) \leq \max_{1 \leq i \leq 4} c(\Pi_i)$, $c(\Pi_1) = 0$, $c(\Pi_2) = 3$, $c(\Pi_3) = 6$, and $c(\Pi_4) = 5$. Theorem 2 follows from the equality $c(\Pi) = 6$.

Now we state the following problem which generalizes essentially Theorem 2 and can be considered as a stronger form of the Sims conjecture.

PROBLEM. Describe the set Π more precisely and find all triples from $\Pi \setminus \Pi_0$ up to equivalency.

The problem is of interest for finite group theory because the study of maximal subgroups is very important for finite group theory. Although local maximal subgroups of finite almost simple groups are now classified, their intersections are not sufficiently investigated. If G is a finite almost simple group and $(G, M_1, M_2) \in$

$\Pi \backslash \Pi_0$, then M_1 and M_2 are some distinct conjugate local maximal subgroups in G whose intersection $M_1 \cap M_2$ is large in a sense (i. e., $c(G, M_1, M_2) > 1$ or $c(G, M_2, M_1) > 1$).

The problem is also of interest for graph theory since the set Π can be used to get a description of undirected connected finite graphs Γ whose automorphism group G acts primitively on $V(\Gamma)$ and $G_x^{[2]} \neq 1$ for $x \in V(\Gamma)$.

The aim of our series of papers is to solve the Problem.

In the first paper of this series (Trudy IMM UrO RAN 20, no. 4 (2014); translation in Proc. Steklov Inst. Math. 289, Suppl. 1 (2015)), we prove the following two theorems.

THEOREM 3 (REDUCTION THEOREM). **If $(G, M_1, M_2) \in \Pi_0$, then $Soc(G) = T^k$, where T is a simple nonabelian group, $k > 1$, and the inequality**

$$c(G, M_1, M_2) \leq c(H, H_1, H_2)$$

holds for some group H such that $Soc(H) \cong T$ and some district conjugate maximal subgroups H_1 and H_2 of H. In particular, $c(\Pi_0) \leq \max_{1 \leq i \leq 4} c(\Pi_i)$.

THEOREM 4. The set Π_1 is empty and, consequently, $c(\Pi_1) = 0$.

In the second paper of the series (Trudy IMM UrO RAN 22, no. 2 (2016); translation in Proc. Steklov Inst. Math. 295, Suppl. 1 (2016))), we prove the following theorem.

THEOREM 5. Let $(G, M_1, M_2) \in \Pi_2$, $Soc(G)$ be a simple group of exceptional Lie type and let $M_1 \cap Soc(G)$ be a non-parabolic subgroup of $Soc(G)$. Then $(M_1, M_2)^3 = (M_2, M_1)^3 = 1$ and one of the following holds:

(a) $G \cong E_6^{\varepsilon}(r)$ **or** $G \cong E_6^{\varepsilon}(r) : 2$,

$\varepsilon \in \{+, -\}$, $r \geq 5$ **is a prime,** $9|(r - \varepsilon 1)$,

$(M_1, M_2)^2 = Z(O_3(M_1))$ **and** $(M_2, M_1)^2 = Z(O_3(M_2))$ **are elementary abelian groups of order** 3^3**,**

$(M_1, M_2)^1 = O_3(M_1)$ **and** $(M_2, M_1)^1 = O_3(M_2)$ **are special groups of order** 3^6**,**

the group $M_1/O_3(M_1)$ **is isomorphic to** $SL_3(3)$ **for** $G \cong E_6^{\varepsilon}(r)$ **and is isomorphic to** $GL_3(3)$ **for** $G \cong E_6^{\varepsilon}(r) : 2$**,**

the group $M_1/O_3(M_1)$ **acts faithfully on** $O_3(M_1)/Z(O_3(M_1))$ **and induces the group** $SL_3(3)$ **on** $Z(O_3(M_1))$**,** $|Z(O_3(M_1)) \cap Z(O_3(M_2))| = 3^2$

and $M_1 \cap M_2 = N_{M_1 \cap Soc(G)}(Z(O_3(M_1)) \cap Z(O_3(M_2)))$**;**

(b) $G \cong Aut({}^3D_4(2))$**,**

$(M_1, M_2)^2 = Z(M_1)$ **and** $(M_2, M_1)^2 = Z(M_2)$ **are groups of order** 3**, not contained in** $Soc(G)$**,**

$M_1 \cong \mathbb{Z}_3 \times ((\mathbb{Z}_3 \times \mathbb{Z}_3) : SL_2(3))$**,**

$(M_1, M_2)^1 = O_3(M_1)$**,** $(M_2, M_1)^1 = O_3(M_2)$

and $M_1 \cap M_2$ **is a Sylow** 3**-subgroup in** M_1**.**

In any case of items (a) **and** (b)**, the triples** (G, M_1, M_2) **from** Π **exist and form one class up to equivalence.**

In the third paper of this series (Trudy IMM UrO RAN 22, no. 4 (2016); translation in Proc. Steklov Inst. Math. 299, Suppl. 1 (2017)), we prove the following theorem.

THEOREM 6. Let $(G, M_1, M_2) \in \Pi_2$**,** $Soc(G)$ **be a simple group of classical non-orthogonal Lie type and let** $M_1 \cap Soc(G)$ **be a non-parabolic subgroup in** $Soc(G)$**. Then** // $(M_1, M_2)^3 = (M_2, M_1)^3 = 1$ **and one of the following holds:**

(a) $G \cong Aut(L_3(3))$**,** $(M_1, M_2)^2 = Z(M_1)$ **and** $(M_2, M_1)^2 = Z((M_2)$ **are groups of order** 2**, non-contained in** $Soc(G)$**,** $M_1 \cong \mathbb{Z}_2 \times S_4$**,** $(M_1, M_2)^1 = O_2(M_1)$**,** $(M_2, M_1)^1 = O_2(M_2)$ **and** $M_1 \cap M_2$ **is a Sylow** 2**-subgroup in** M_1**;**

(b) $G \cong U_3(8) : 3_1$ **or** $U_3(8) : 6$**,** $(M_1, M_2)^2 = Z(M_1)$ **и** $(M_2, M_1)^2 = Z(M_2)$ **are groups of order** 3**, not contained**

in $Soc(G)$**,** $M_1 \cong \mathbb{Z}_3 \times (\mathbb{Z}_3^2 : SL_2(3))$ **or** $\mathbb{Z}_3 \times (\mathbb{Z}_3^2 : GL_2(3))$ $(M_1, M_2)^1 = O_3(M_1)$, $(M_2, M_1)^1 = O_3(M_2)$ **and** $M_1 \cap M_2$ **is a Sylow** 3**-subgroup in** M_1 **or its normalizer in** M_1**, respectively;**

(c) $G \cong L_4(3) : 2_2$ **or** $Aut(L_4(3))$, $(M_1, M_2)^2 = Z(M_1)$ **и** $(M_2, M_1)^2 = Z(M_2)$ **are groups of order** 2**, not contained in** $Soc(G)$**,** $M_1 \cong \mathbb{Z}_2 \times S_4 \times S_4$ **or** $\mathbb{Z}_2 \times (S_4 \wr \mathbb{Z}_2)$**, respectively,** $(M_1, M_2)^1 = O_2(M_1)$, $(M_2, M_1)^1 = O_2(M_2)$ **and** $M_1 \cap M_2$ **is a Sylow** 2**-subgroup in** M_1**.**

In any case of items (a), (b) **and** (c)**, the triples** (G, M_1, M_2) **from** Π **exist and form one class up to equivalence.**

The description of Π_2 will be completed at our fourth paper, which is in preparation. In particular, the following theorem is proved.

THEOREM 7. Let $(G, M_1, M_2) \in \Pi_2$, $Soc(G)$ **be a simple orthogonal group of the dimension** ≥ 7 **and** $M_1 \cap Soc(G)$ **be a non-parabolic subgroup in** $Soc(G)$**. Then** $Soc(G) \cong P\Omega_8^+(q)$**, where** q **is a prime power. Moreover if** q **is an odd prime, 16 divides** $q^2 - 1$, G **is a finite group with** $Soc(G) \cong P\Omega_8^+(q)$ **and** G **contains an element inducing on** $Soc(G)$ **a graph automorphism of order 3 (so-called triality) then there exists a triple** (G, M_1, M_2) **from** Π_2 **such that** $(M_1, M_2)^2 = Z(O_2(M_1))$ **and** $(M_2, M_1)^2 = Z(O_2(M_2))$ **are elementary abelian groups of order** 2^3, $(M_1, M_2)^1 = O_2(M_1)$ **and** $(M_2, M_1)^1 = O_2(M_2)$ **are special groups of order** 2^9**, the group** $M_1/O_2(M_1)$ **is isomorphic to** $PSL_3(2) \times \mathbb{Z}_3$ **or** $PSL_3(2) \times S_3$**, and** $M_1 \cap M_2$ **is a Sylow** 2**-subgroup in** M_1**.**

In subsequent papers of the series, the sets Π_3 and Π_4 will be described.